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INSTABILITY OF THE CONSERVATIVE PROPERTY UNDER QUASI-ISOMETRIES

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Abstract

The paper shows by example the existence of a pair of quasi-isometrically equivalent complete Riemannian metrics g, $\tilde{g}$ on a manifold N so that (N, g) is conservative while $(N, \tilde{g})$ is not. This contrasts with known sufficient criteria for conservativeness which are stable under such changes in metric. In doing so we also give an example of a complete manifold with no nonconstant bounded solutions to Laplace's equation but with nonconstant bounded solutions to the heat equation.

1. Introduction

A smooth Riemannian manifold (N, g) has associated with it a number of standard and interrelated geometric objects. In particular, it possesses a unique geodesic through each point in each tangent direction, a Laplace Beltrami operator, and a Markovian family $(\mathbf{P}_n, n \in N)$ of measures on the continuous paths called Brownian motion. The measures $\mathbf{P}_n$ can be constructed as the limit where $\varepsilon \to 0$ of the simple random walk one obtains by: starting at n, using the metric to choose a direction uniformly at random, moving at speed $\frac{1}{\varepsilon}$ along the geodesic a distance $\sqrt{\varepsilon}$, and then repeating the process with this new point as starting point. The law of this process is a probability measure on paths in N; the weak limit $\mathbf{P}_n$ is Brownian motion. The semigroup P_t defined by $P_t f(n) = \mathbf{E}_n(f(X_t))$ has $\frac{1}{2}\Delta$ as the infinitesimal generator, where $\mathbf{E}_n$ is the expectation or integral against $\mathbf{P}_n$.

As the example of the usual disc $D = \{|z| < 1\} \subset \mathbf{C}$ shows, geodesics do not need to stay in N for all t and might explode (that is, eventually leave every compact set) in finite time. It is a well-known piece of analysis that any geodesic can be extended either until it explodes or for all time; oscilliatory discontinuities do not occur. If all geodesics extend for all

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time we say N is complete, and this definition is compatible with the usual metric space one.

In an analogous way, the example of the disc shows that Brownian motion X_t can leave a manifold in finite time with probability one. Just as in the case of deterministic geodesics, there is a dichotomy: with probability one, either a path X_t explodes (leaving every compact set for a last time) at time $\xi < \infty$, or it is defined as a continuous path in N for all t (in which case we say $\xi = \infty$). In this case $\lim_{t\to\infty} X_t$ does not exist in N.

In the case where $\xi = \infty$ almost surely we say (N, g) is *stochastically complete* (or conservative). This property has a number of equivalent purely analytic forms (see [1]).

The punctured complex plane illustrates that a manifold can be stochastically complete without being complete. A suitably chosen conformal and rotation invariant change of metric on the disc demonstrates that the converse can also occur. However, an unublished manuscript of Li and Karp, [4] and a more recent paper of Grigoryan [2] and preprint by Takeda [7] all give reasonably sharp (and quasi-isometry invariant) criteria for stochastic completeness of complete Riemannian manifolds. In particular, a complete Riemannian manifold will be stochastically complete if as a function of radius r the volume of a ball centred on some fixed point grows more slowly than e^{r^α} for some $\alpha \leq 2$.

Varopoulos has shown [8] that a subquadratic bound on the curvature would always ensure nonexplosion. This paper is concerned with the stability of stochastic completeness.

For example, if (N, g) is a complete and stochastically complete manifold and $\tilde{g}$ is a uniformly equivalent metric on N (so that the supremum over all $n \in N$ and $u \in TN_n \setminus \{0\}$ of $|\log(g(u, u)/\tilde{g}(u, u))|$ is finite), then will $(N, \tilde{g})$ be complete and stochastically complete? The answer to the former is obviously yes because of the uniform equivalence of the metric spaces. This paper shows that $(N, \tilde{g})$ need not in general be stochastically complete.

The idea is relatively simple, the first step will be to look at a surface with two metrics $(N, g, \tilde{g})$ and understand the two diffusions X and $\widetilde{X}$. The manifolds (N, g) and $(N, \tilde{g})$ will actually both have constant negative curvature -1 and locally uniform geometries. Neither X nor $\widetilde{X}$ explodes.

The second stage will be to introduce a conformal change to the metrics $h = sg$ and $\tilde{h} = s\tilde{g}$, where s is a positive scalar function. Of course h and $\tilde{h}$ are equivalent if g and $\tilde{g}$ were, and because (N, g) is a Riemannian *surface* the new Brownian motion Y on (N, h) is a simple time change of the old one. Put

$$\tau(t) = \int_0^t s(X_{t'})\,dt' \quad \text{and} \quad Y_\tau = X_t;$$

then Levy observed that Y is Brownian motion on (N, h) if X was Brownian motion on (N, g). Clearly (N, h) will be stochastically complete if and only if $\int_0^\infty s(X_{t'})\,dt' = \infty$.

In what follows we will generally use hyperbolic two-space [6] as a reference manifold which will induce geometries on other manifolds, and it would be possible to construct a suitable s directly on $\mathbf{H}$. However, it will be conceptually easier to work on a manifold which "looks like" the Caley graph of the free group Γ on two generators—as the tree structure will simplify matters. Most of the technical work has already been done in [5].

2. The two metrics on N

Consider Γ, the free group generated by the examples a, b without relations. Any element $\gamma \in \Gamma$ has a unique shortest expression as a product of the generators a, b and their inverses a^{-1}, b^{-1} called the reduced word for γ. Let $|\gamma|$ denote the length of this reduced word, with $|\gamma|_a$ the number of $a^{\pm 1}$'s and $|\gamma|_b$ the number of $b^{\pm 1}$'s in it. Clearly $|\gamma| = |\gamma|_a + |\gamma|_b$.

Introduce a family of reversible random walks on Γ defined by

$$\mathbf{P}^p(X_{n+1} = gX_n) = \begin{cases} p/2 & \text{if } g = a^{\pm 1}, \\ (1-p)/2 & \text{if } g = b^{\pm 1}, \\ 0 & \text{otherwise}, \end{cases}$$

where $p \in (0, 1)$ parameterizes the family.

Let G be the Caley graph with vertex set Γ, so γ is connected to γ' if $|\gamma'\gamma^{-1}| = 1$; the edge is an "a" edge if $\gamma'\gamma^{-1} \in \{a, a^{-1}\}$, otherwise it is a "$b$" edge. Attach to each a-edge the weight $w_{\gamma\gamma'} = p$ and each b-edge the weight $w_{\gamma\gamma'} = 1 - p$. Then, if p, p' come from a compact subset of $(0, 1)$ we have $\frac{1}{c} < w^p_{\gamma\gamma'}/w^{p'}_{\gamma\gamma'} < c$ uniformly over edges $\gamma\gamma'$. The weights can be used in the usual way to construct $\mathbf{P}^p$ as a reversible Markov chain in the usual way (see [5, §1]).

As X progresses, the $\mathbf{P}^p$-probability that $|X_{n+1}| - |X_n| = +1$ is at least $\rho = \min[(1 - p/2), 1 - ((1 - p)/2)]$, where $\rho > \frac{1}{2}$, otherwise

$||X_{n+1}|-|X_n||<1$. Thus, $\lim_{n\to\infty}|X_n|/n \geq (p-\frac{1}{2})$, and in fact the Kingman subadditive ergodic theorem [3] implies that $n^{-1}|X_n|_a$ and $n^{-1}|X_n|_b$ converge as n goes to ∞.

In [5, §3] this author demonstrated that ε and δ exist so that if $p<\varepsilon$

$$\lim_{n\to\infty}\frac{|X_n|_a}{|X_n|}<\frac{1}{2+\varepsilon},$$

and conversely.

In [5, §§4, 5] it was also explained how to construct a manifold L which was a spheres and tubes approximation to the graph G, and introduce a family of conformal structures on K so that any compatible metric g^p and associated Brownian motion Z_t^p on L had the property that each time when it went from a sphere S_γ to another sphere $S_{\gamma'}$, $|\gamma'\gamma^{-1}|=1$, and moreover the following hold:

(i) $\mathbf{P}(|\gamma'|>|\gamma|)>p>\frac{1}{2}$ uniformly in γ.

(ii) Denote that part of N made up of spheres S_γ with $|\gamma|_a > (1+\varepsilon)|\gamma_b|$ together with adjoining tubes by A. Then A is an absorbing set for the diffusion Z_t^p whenever p is close enough to 1.

(iii) Γ acts in a natural way on L, and all the conformal structures on L lifted from L/Γ are invariant.

In fact, L/Γ can be taken to be the sphere with two handles, and g_p is the lift of the canonical hyperbolic metric on K.

Now the ergodic theorem for Brownian motion on the compact manifold K guarantees that the number of crossings Z_t makes from one sphere S_γ to an adjacent one $S_{\gamma'}$ is approximately linear in time, and so we can deduce that $\lim_{t\to\infty} d^p(Z_0^p, Z_t^p)/t > (p\cdot\frac{1}{2})>0$, where d is the Riemannian distance. If one lifts the process up the universal cover $\mathbf{H}$, one obtains ordinary hyperbolic Brownian motion Z^h. Standard properties of Brownian motion there ensure the inequality

$$\overline{\lim_{T\to\infty}}\frac{d(Z_T^h, Z_0^h)}{T}<\infty,$$

where d is hyperbolic distance, and Z^h (the lift of Z^p) is hyperbolic Brownian motion (see, for example, [6]).

3. The conformal transformation

Consider L a bit further. Let g, $\tilde{g}$ be two metrics on L corresponding to g^p for p close to zero, and again for p close to 1. Because g, $\tilde{g}$ are defined on $L/\Gamma = K$ and K is compact, they are uniformly equivalent. Clearly they are complete metrics, for let $\rho(t)$ be any geodesic in (L, g) that explodes; the only way it can do this is by visiting ∞-many spheres. Let $S_{\gamma_1} S_{\gamma_2}, \cdots$ be the sequence of distinct spheres visited. Then $|\gamma_j \gamma_{j-1}^{-1}| = 1$ for all j, and the minimum distance from one sphere S_γ to one of four neighbours is some strictly positive ε, independent of γ so the length of the geodesic ρ is at least $n\varepsilon$ by the time it has visited the nth sphere in the sequence, and thus has length infinity. As it travels at unit speed it is defined for all time.

Which conformal changes of metric preserve the completeness? Consider (L, sg); suppose that s is slowly varying in that there is a $C > 1$ such that uniformly in γ if x, x' are in S_γ, the directly neighbouring spheres $S_{\gamma'}$, or the connecting cylinders, then

$$\frac{1}{C} < \frac{s(x)}{s(x')} < C.$$

Fix $0 \in S_e$, and let $x_\gamma = \gamma 0$; then

$$d_{sg}(S_\gamma, S_{\gamma'}) \geq \left[\frac{S(x_\gamma)}{C}\right]^{1/2} d_g(S_\gamma, S_{\gamma'}),$$

whenever γ, γ' are adjacent.

Lemma. *Suppose s is slowly varying and $s(x_\gamma) > \psi(|\gamma|)$, where $\sum_{n=1}^\infty [\psi(n)]^{1/2} = \infty$; then (L, sg) is complete.*

The proof is as before. Any geodesic going to infinity must for some N and for each $n > N$ pass from some S_γ to $S_{\gamma'}$ with $|\gamma| = n$ and $|\gamma\gamma'^{-1}| = 1$. It does this in distinct time intervals. It follows that a lower bound for its length is $C^{-1/2} \sum_N^\infty \psi(n)^{1/2}$. q.e.d.

In particular, it suffices that $s(x) \sim d_g(0, x)^\alpha$ for any $\alpha \geq -2$, for the space to be complete.

Now what sort of choice for s will ensure stochastic (in)completeness? One uses the fact that $d_g(0, Z_t) \sim t$ to see that $\int_0^\infty s(Z_t)\,dt < \infty$ if $s(x) \sim d_g(0, x)^\beta$, where $\beta < -1$, and $\int_0^\infty s(Z_t)\,dt = \infty$ if $\beta \geq -1$.

Lemma. *Let $A \subset L$ be an absorbing set for Z and suppose that on A*

(i) $s(x) \sim d(0, x)^\beta$, $\beta \geq -1$, *or*

(ii) $s(x) \sim d(0, x)^\beta$, $\beta < -1$.

Then (L, sg) is stochastically (i) *complete or* (ii) *incomplete.*

Proof. The divergence or convergence of $\int_0^\infty s(X_t)\,dt$ only depends on what happens near ∞, and as $X_t \in A$ for all large t, $s(X_t) \sim t^\beta$ for all large t.

4. The final example

It is now absolutely straightforward to construct the complete example.

Let $A \subset L$ be the union of those spheres S_γ with their joining tubes for which $|\gamma|_a > (\frac{1}{2} + \varepsilon)|\gamma|$, and let B be the similar region where $|\gamma|_b > (\frac{1}{2} + \varepsilon)|\gamma|$. A is absorbing for Z_t, B for $\widetilde{Z}_t$. If $s - d(0, x)^{-\beta}$ on A and $s \sim 1$ on B, where $1 < \beta \leq 2$, then (L, g) and $(L, \tilde{g})$ are both complete, however only $(L, \tilde{g})$ is stochastically complete. A suitable choice for s would be some interpolation of the function which at x_γ has the value

$$\{(|\gamma|_a - |\gamma|_b)^+ + 1\}^{-\beta}, \qquad \beta \in (1, 2],$$

and in fact $p = \frac{1}{2}$ would be the critical point in this case. No one has investigated the behaviour at this point as β changes.

Final remark. A number of years ago a colleague asked if one could find a complete manifold with no nonconstant bounded (positive) harmonic functions but lots of bounded solutions to the heat equation; at that time this author could not. The approach here provides an example. In [5, §6] we constructed a more complex manifold modelled on the free group on two generators which had no positive harmonic functions. The same conformal metric change can be applied there; and precisely because the extra links were chosen to connect only words with the same proportion of a's and b's one discovers that the same function s will introduce a conformal change of metric which preserves completeness but forces explosion. Explosion gives a very strong existence results for globally defined bounded nonconstant solutions to the heat equation on $L \times \mathbf{R}_+$.

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